Simulation of Size Reduction and Enlargement Processes by a Modified Version of the Beta Distribution Function

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Most of the distribution functions commonly used in the description of particulate size distributions (Herdan, 1960; Allen, 1981), notably the log normal distribution, have an infinite range and/or a spread that is dependent on the mode. In contrast, real particle populations have a finite size range determined by physical considerations. Moreover, the application of a function of the log normal distribution type requires that any process that affects the population mean size or mode must affect the distribution spread in a predetermined manner. Needless to say, one can envision many processes in which the population mode and spread can vary independently, or almost independently, as a result of the process conditions. This is particularly the case in attrition or agglomeration processes where the skew of the distribution can shift from right to left or vice versa. This communication demonstrates that by using a slightly modified version of the beta distribution function, all these difficulties can be avoided.

Modified Beta Distribution

The common presentation of the beta function, $\beta(x)$, is in the form (Lukacs, 1960):

$$\beta(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} x^{p-1}(1-x)^{q-1} \tag{1}$$

It has the range $0 < x < 1$ and $\beta(0) = \beta(1) = 0$.

It can be applied to any finite range by the substitution:

$$x = \frac{Z - Z_{\min}}{Z_{\max} + Z_{\min}} \tag{2}$$

where the Z's represent the actual size in length units. The spread depends on both coefficients p and q, and so does the mode, x_m. The latter is determined from the condition that $d\beta(x)/dx = 0$ or

$$x_m = \frac{p-1}{p+q-2} \tag{3}$$

The interdependency between mode and spread can be eliminated if Eq. 1 is rewritten in the form

$$\beta(x) = \frac{\Gamma[m(a+1)+2]}{\Gamma(am+1)\Gamma(m+1)} x^{am}(1-x)^{m} \tag{4}$$

or alternatively with $x^m(1-x)^{am}$ instead of $x^{am}(1-x)^m$. In this form the spread is determined solely by the coefficient m, and the mode x_m by the coefficient a, i.e.,

$$x_m = \frac{a}{a+1} \tag{5}$$

It is easy to show that in this format, skew to the right is expressed by $a > 1$, skew to the left by $a < 1$, and a symmetric distribution by $a = 1$.

Manual calculation of $\beta(x)$ from Eq. 4 (or Eq. 1) is a cumbersome task, particularly when the coefficients are not integers. The difficulty can be overcome, however, if a computer is used. The program can be based on direct calculation of the terms that contain the gamma function, a convenient option when a and m are integers, or by employing the raw version of the distribution function $f_{am}(x)$ using a numerical integration procedure, i.e.,

$$f_{am}(x) = \frac{x^{am}(1-x)^m}{\int_0^1 x^{am}(1-x)^m dx} \tag{6}$$

It should be mentioned that numerical integration procedures, such as those based on Romberg's method, are easily implemented on a personal computer and therefore using Eq. 6 is almost equivalent to using Eq. 4 except for prolongation of the calculation time.

Simulation of Changes in Particle Size Distributions

Since the distribution mode and spread can be varied independently, particle size changes in growth or reduction processes can be simulated by assigning appropriate time and/or other dependencies to Eq. 6 (or Eq. 5) coefficients. An example of a simulated batch size reduction operation performed with an Apple IIe computer is shown in Figure 1. The (arbitrary) assumptions were that the mode decreases exponentially with time and that the distribution has a maximum spread, when the mode is half the range. This translates to the following expressions.

$$x_m = \frac{a}{a+1} = x_{mo} \exp - \left(\frac{t}{\tau}\right) \tag{7}$$

$$a = \frac{x_{mo} \exp - \left(\frac{t}{\tau}\right)}{1 - x_{mo} \exp - \left(\frac{t}{\tau}\right)} \tag{8}$$

and

$$m = \frac{B(a+1)}{a} \tag{9}$$

where x_{mo} is the initial normalized mode, τ is a time characteristic, and B is a coefficient derived from the empirical expression.

$$m = \frac{B}{x_m(1 - x_m)} \tag{10}$$

which describes a function with minimum m (largest spread) at $x = 1/2$.

In this format the whole process is characterized by only three constants, namely, x_{mo}, τ, and B. It ought to be mentioned that the selection of the expressions for describing the temporal relationship of the mode and spread measures was based solely on mathematical simplicity considerations and not on any kinetic mechanism. Therefore, expressions of the kind described in this communication ought to be considered as purely phenomenological models whose main advantage is mathematical convenience and a minimal number of constants. Similar simulations can be created for size enlargement operations, e.g., agglomeration. An example is shown in Figure 2. In this case the assumption was that the size increase can be described by:

$$x_m = \frac{a}{a+1} = x_{mo} + (x_{mas} - x_{mo})\left[1 - \exp - \left(\frac{t}{\tau}\right)\right] \tag{11}$$

or

$$a = \frac{x_{mas} - (x_{mas} - x_{mo}) \exp - \left(\frac{t}{\tau}\right)}{1 - x_{mas} + (x_{mas} - x_{mo}) \exp - \left(\frac{t}{\tau}\right)} \tag{12}$$

where x_{mas} is the asymptotic mode and τ is a time constant. The spread measure m was described by the expression given in Eqs. 9 and 10, i.e., assuming maximum spread when $x = 1/2$.

More elaborate models can easily be developed to more

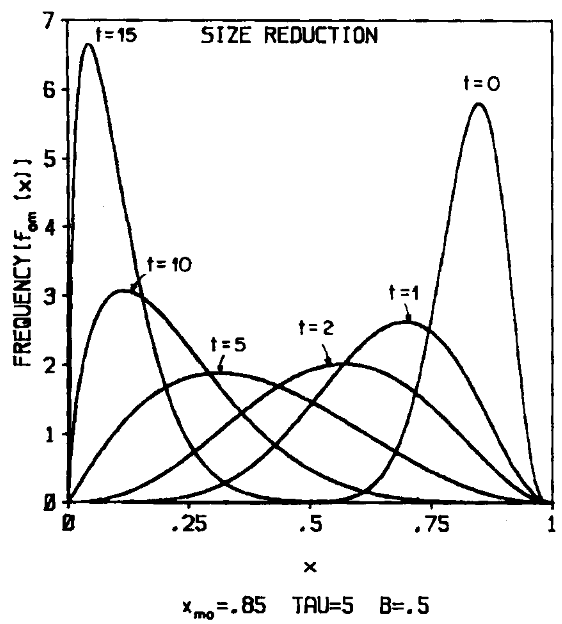

Figure 1. Simulated size distributions in a hypothetical size reduction process based on Eqs. 6, 8, and 9.

The format of Eqs. 8 and 9 as well as the magnitude of the parameters was arbitrarily selected.

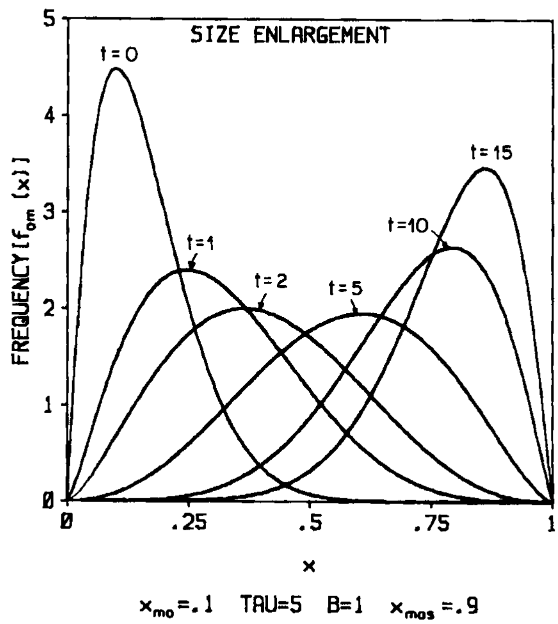

Figure 2. Simulated size distributions in a hypothetical size enlargement process based on Eqs. 6, 9, and 12.

The format of Eqs. 9 and 12 as well as the magnitude of the constants was arbitrarily selected.

closely resemble real processes. Similarly, experimental distribution curves can be fitted by the proposed model format using nonliner regression programs, thus enabling the determination of the model constants and their temporal dependencies in a more realistic manner. It should also be noted that many size reduction and enlargement operations involve mixed distributions (i.e., multimodal distributions). For these, the described simulations are certainly inappropriate. The methodology, however, can be extended to such problems using the structure of Eq. 6 but with more terms.

Acknowledgment

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Notation

a = power coefficient
B = constant
f_{am} = distribution (density) function
m = power coefficient
p = power coefficient
q = power coefficient
t = time
x = normalized size
x_m = mode
x_{mas} = asymptotic mode
x_{mo} = initial mode
Z = absolute particle size, length units
Z_{min} = smallest absolute particle size, length units
Z_{max} = largest absolute particle size, length units

Greek letters

β = beta distribution (density) function
Γ = gamma function
τ = time constant, time units

Literature cited

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